Pergamon

www.elsevier.com/locate/jappmathmech

PII: S0021-8928(01)00071-5

J. Appl. Maths Mechs, Vol. 65, No. 4, pp. 671–679, 2001

0021-8928/01/$—see front matter

AN ENERGY ANALYSIS OF THE PRESCRIBED MOTION OF AN OSCILLATOR†

W. SCHIEHLEN

Stuttgart, Germany

e-mail: ws@mechb.uni-stuttgart.de

(*Received* 3 *January* 2001)

The energy consumption in the problem of the prescribed periodic motion of an oscillator is analysed. Since the solution of this problem by actuators involves extra weight and hence additional energy is required, it is proposed to use springs with linear or non-linear characteristics for local energy storage. The results obtained enable the optimal choice of the spring parameters to be made.

It is will known that control principles based on inverse dynamics unable high non-linearities, typical for mechanical systems undergoing large displacement motion [1–7], to be overcome. Based on an accurate model of the system under consideration, all the non-linearities are first compensated by control action and the remaining double integrator is then controlled by linear feedback. This approach is very attractive to control engineers since a wide variety of design tools can be used successfully.

In previous publications on control engineering, inverse dynamics is considered as a control problem only related to date processing, and the required energy is assumed to be available. This is certainly the case for most systems which are stationary but is no longer true for autonomous robots and walking machines.‡ Some important thoughts on the design of power systems for legged vehicles were presented in [8], where it was pointed out that the missing power regeneration capability of most electric and hydraulic actuators boosts energy consumption. Furthermore, the energy lost generates large amounts of heat which may be difficult to remove from the system components.

The theory of constrained mechanical systems, an analysis of their stability and a computational evaluation have been given, in particular, in [9–12]. A survey of multibody dynamics was presented in [13], and power aspects were also considered in [14].

1. AN ACTIVELY CONTROLLED HARMONIC OSCILLATOR

It is well known that a harmonic oscillator is characterized by a fixed frequency which does not depend on the initial conditions. In technological processes, however, motions with an adjustable frequency are often required, resulting in active control. Due to the necessary power supply the controlled harmonic oscillator is no longer conservative. In this section an actively controlled oscillator supplemented by a passive spring is considered (Fig. 1). Such a system is sufficiently simple to unable straightforward computations to be made without complex computer techniques.

The equation of motion is

$$m\ddot{x}(t) + cx(t) = u(t) \tag{1.1}$$

where m is the mass, c is the spring coefficient and $u(t)$ is the actuator force. The actuator may be fully active, providing acceleration and deceleration forces, or semi-active, generating only acceleration forces by propulsion while the deceleration forces are due to an additional braking system. It is assumed that the fully active actuator features a direct drive without any gears and the braking system does not require any power for its operation.

Depending on the technological process the oscillator has to move with a prescribed amplitude A and frequency ω

†*Prikl. Mat. Mekh.* Vol. 65, No. 4, pp. 688–697, 2001.

‡HONDA Motor Co., Ltd.: Humanoid Robot – Specification. May 2000. URL:http://www.honda.co.jp/english/technology/robot/spec1.html

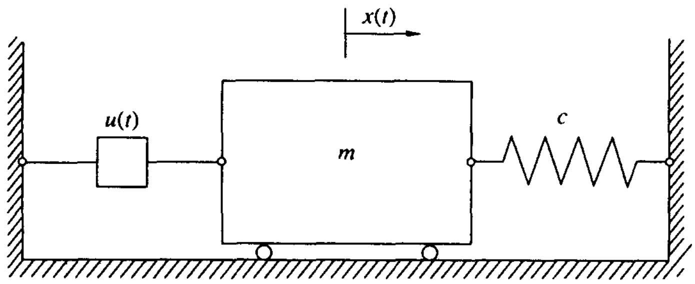

Fig. 1

$$x(t) = -A \cos \omega t \tag{1.2}$$

which implies a rheonomic constraint for the moving mass. The actuator force follows from the principle of inverse dynamics with (1.1) as

$$u(t) = A(m\omega^2 - c) \cos \omega t \tag{1.3}$$

The power required is given by

$$P(t) = u(t)\dot{x}(t) = \tfrac{1}{2} A^2 (m\omega^2 - c)\omega \sin 2\omega t \tag{1.4}$$

Obviously, no power is required when $c = m\omega^2$, i.e. if the spring coefficient c is adjusted to the frequency ω of the prescribed motion. In all other cases there is a periodic oscillation of the power with frequency 2ω. In particular, positive power means acceleration and negative power means deceleration enforced by the actuator. For $c < m\omega^2$ (a weak spring), the actuator has to push the mass forward in the first quarter-cycle, while for $c > m\omega^2$ (a strong spring), the actuator has to hold back the mass at the beginning of the cycle.

The major and more-important question is where the power is coming from and how much energy is consumed. In any case the energy consumption has to be paid for.

The energy, or more generally speaking the work, is the time integral of the power

$$W(t) = \int_0^t P(\tau) d\tau \tag{1.5}$$

Using (2.4) this yields

$$W(t) = \tfrac{1}{4} A^2 (m\omega^2 - c)(1 - \cos 2\omega t) \tag{1.6}$$

However, this equation is only true, if the deceleration energy can be stored. An actively controlled actuator does not have any energy capabilities and, therefore, the deceleration energy has to be provided or, at least, converted into heat.

The intervals of positive and negative power must be treated separately. For this purpose, first the time t_i ($i = 1, \dots, n$) has to be found where the power vanishes $P(t_i) = 0$. Then, the element of work for the interval $t \in [t_{i-1}, t_i]$ has the form

$$\Delta W_i(t) = \int_{t_{i-1}}^{t_i} P(\tau) d\tau \tag{1.7}$$

(with $t_0 = 0$). Finally, the total work is obtained by summation, according to the different kinds of actuators. For this purpose the following function is defined

$$W_i(t) = \begin{cases} 0, & \text{if} \quad t < t_{i-1} \\ \Delta W_i(t), & \text{if} \quad t_{i-1} < t < t_i \\ \Delta W_i(t_i), & \text{if} \quad t > t_i \end{cases} \tag{1.8}$$

We then obtain the total work (from now on the summation limits are from $i = 1$ to $i = n$)

$$W(t) = \begin{cases} \sum |W_i(t)| & \text{for a fully active actuator} \\ \tfrac{1}{2}\sum[1 + \text{sign}\, W_i(t)]\, |W_i(t)| & \text{for a semi-active actuator} \end{cases} \tag{1.9}$$

(in the case of the semi-active actuator only the positive contributions to the total work due to acceleration intervals are summed).

For harmonic oscillations, the zeros of function (1.4) are $t_i = [\pi/(2\omega)]i$, and from (1.7) we have

$$\Delta W_j(t) = (-1)^{j+1} A^2 (m\omega^2 - c)(1 - (-1)^{j+1} \cos 2\omega t)/4 \tag{1.10}$$

A graphical representation of the work is shown in Fig. 2 for two intervals and a weak spring ($c < m\omega^2$). Obviously W_3 represents acceleration work and W_4 characterizes deceleration work. The total work for the different kinds of actuators follows from (1.10) and is shown in Fig. 3 for a system without a spring ($c = 0$). Here curves 1 and 2 correspond to a fully active actuator without spring and a semi-active actuator without a spring, respectively.

For comparison, a passive system with $c \neq 0, u(t) = 0$ is considered. Then, the power, total work and kinetic energy have the form

$$P(t) = -cx(t)\dot{x}(t) = \tfrac{1}{2} A^2 c\omega \sin 2\omega t, \quad W(t) = \tfrac{1}{4} A^2 c(1 - \cos 2\omega t)$$

$$E(t) = \tfrac{1}{2} m\dot{x}^2(t) = \tfrac{1}{2} A^2 m\omega^2 \sin^2 \omega t = W(t) \tag{1.11}$$

On the other hand, with notation (1.8) for the case under discussion the total work can be represented as the following sum

$$W(t) = \sum W_i(t)$$

In the case of a conservative system the deceleration energy is completely stored and no energy is consumed. The oscillation of the energy is also shown for the passive system in Fig. 3 (line 3).

The energy consumed by the actively controlled oscillator increases linearly with time. The growth rate can be obtained from the mean value of the power in a quarter cycle or a half cycle depending on the type of actuator. For the fully-active actuator it remains

$$\overline{P}_1 = \frac{2\omega}{\pi} \frac{1}{2} A^2 \omega (m\omega^2 - c) \int_0^{\pi/(2\omega)} \sin 2\omega t dt = \frac{1}{\pi} A^2 \omega (m\omega^2 - c)$$

and for the semi-active actuator, averaged over a half cycle, we have

$$\overline{P}_2 = \frac{1}{2\pi} A^2 \omega (m\omega^2 - c)$$

The linear increase in the energy consumed is finally

$$\overline{W}(t) = |\overline{P}|\, t$$

where the fully active oscillator is twice as expensive as the semi-active actuator (the dashed lines in

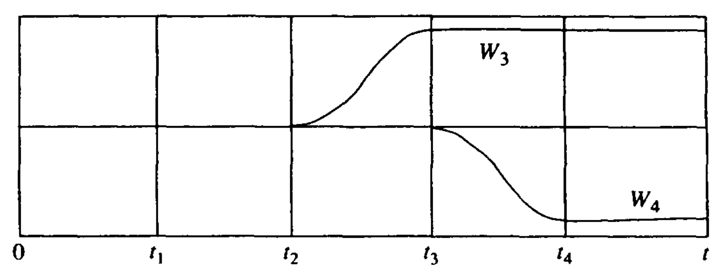

Fig. 2

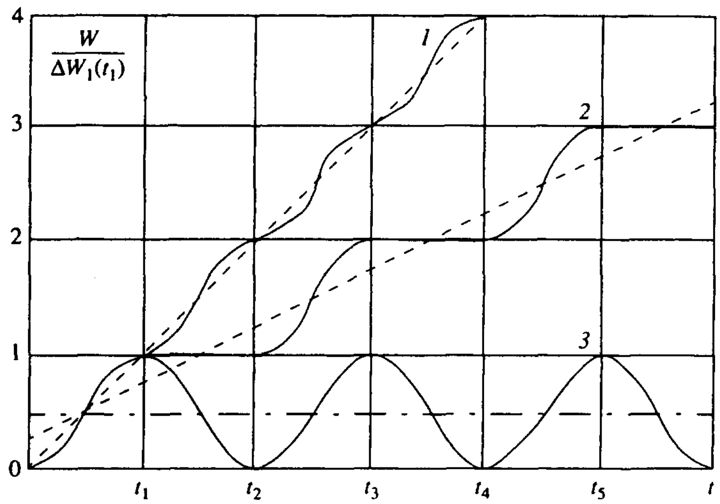

Fig. 3

Fig. 3). As a matter of fact, the maximum power required and the growth rate of energy consumption depend on the function

$$f(\omega)=|\,\omega(m\omega^2-c)\,| \tag{1.12}$$

which will be discussed in more detail. In technological processes there is always an upper limit ω_2 and there may be a lower limit $\omega_1 \geqslant 0$ also. Thus, the discussion of the function $f(\omega)$ can be restricted to the given frequency limits $\omega \in [\omega_1, \omega_2]$. The aim is to reduce the maximum power and the growth rate of energy consumption by using the storage capacity of an elastic spring.

When there is no spring ($c = 0$) we obtain the maximum value $f(\omega_2) = m\omega_2^3$, which can be reduced to zero by a spring with $c = m\omega_2^2$. Then, for lower frequencies the spring is too strong and the actuator has to decelerate in the first time interval. However, the maximum value of the function f when $\omega = \omega_2/\sqrt{3}$ is considerably less

$$f(\omega_2/\sqrt{3}) = 2m\omega_2^3/(3\sqrt{3})$$

which means a reduction in the energy consumption of 62%. A further reduction can be achieved by using a weaker spring. Let $c = \varkappa m\omega_2^2$, $\varkappa \in (0, 1)$. Then the function (1.12) has a unique critical point $\omega^* = (\varkappa/3)^{1/2}\omega_2$, while the quantity $f(\omega^*) = 2(\varkappa/3)^{3/2}\omega_2^3$ increases monotonically with $\varkappa$. On the other hand, the boundary value $f(\omega_2) = (1 - \varkappa)\omega_2^3$ decreases monotonically with $\varkappa$. Therefore, the maximum value of function (1.12) is a minimum when $f(\omega^*) = f(\omega_2)$, i.e. $\varkappa = 3/4$, and the maximum value equals

$$f(\omega_2/2) = m\omega_2^2/4$$

Thus, a further reduction of 13% in the energy consumption is achieved, resulting in a total reduction total of 75%. All three cases are shown in Fig. 4.

It should be noted that the optimal solution results not only in the lowest energy consumption but also in the most uniform power distribution. While the energy consumption is related to the operating costs, the power supply involves investment costs. Therefore, a uniform power distribution with respect to the frequency variations is of great economic interest also.

Additional information on the statistics of the frequency variation offers the possibility of a further adjustment of the energy storage spring but this is beyond the scope of this paper.

Small frequency variations can be analysed more easily. For $\omega = \omega_0 \pm \Delta\omega$, $\Delta\omega \ll \omega_0$ the spring coefficient is chosen to be $c = m\omega_0^2$. Then, the function (1.12) exhibits a linear behaviour in a neighbourhood of ω_0, $f(\omega) = 2m\omega_2^0\,|\Delta\omega|$, and the maximum energy consumption depends linearly on the maximum frequency variation. However, the energy consumption and power requirements remain small, most of the energy being stored in the spring. However, the nominal frequency ω_0 is the most important parameter.

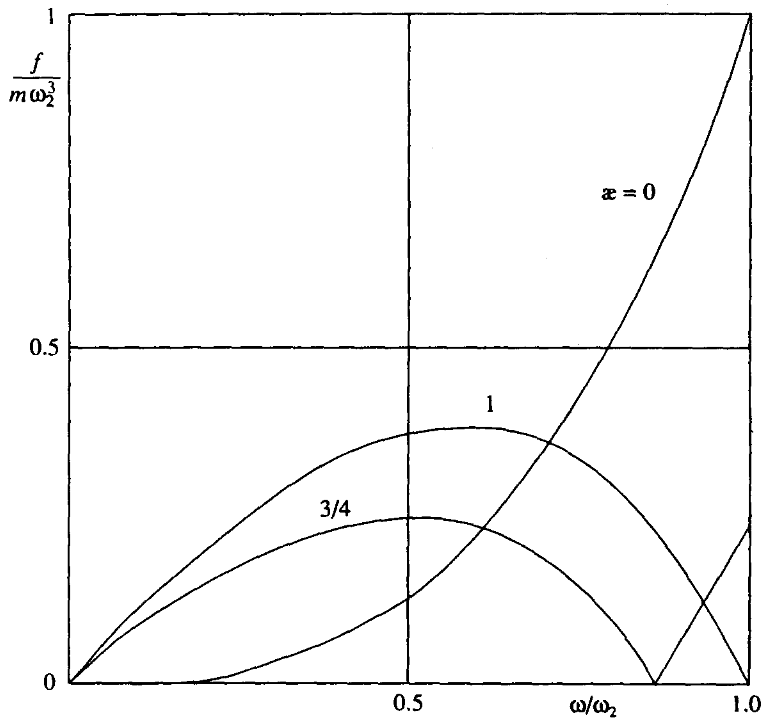

Fig. 4

To summarize, an actively controlled linear oscillator, unlike a passive linear oscillator, enables motions with arbitrary frequencies to be paid for by some energy consumption. An elastic spring is a perfect device for energy storage and even for very large variations of the technologically prescribed frequency, resulting in energy and power savings of up to 75%. Similar results can also be obtained for an actively driven pendulum and have been published in [9] in connection with the design of power systems for legged vehicles.

2. AN OSCILLATOR WITH AN ARBITRARY PRESCRIBED MOTION AND A LINEAR SPRING

The harmonic motions considered above do not always meet the requirements of technological processes. An arbitrary periodic motion may be prescribed or the trajectories may have to be optimal, resulting in a switching control, respectively. In addition, the period of the motion may change depending on the technological process.

Consider again the simple mechanical oscillator with a linear spring (1.1) shown in Fig. 1. The technological process is time optimal with constant acceleration and deceleration intervals. Then, in the first quarter of the first cycle it yields

$$x(t) = -A + at^2/2, \quad 0 \leqslant t \leqslant t^*, \quad t^* = \sqrt{2A/a} = \pi/(2\omega) \tag{2.1}$$

where A is the amplitude, $a > 0$ is the constant acceleration, and ω is the frequency of the process. The parameters a and ω are related by the equality in the last formula of (2.1), i.e. only one of them can be chosen.

The actuator force follows from the principle of inverse dynamics with (1.1) as

$$u(t) = ma - cA + cat^2/2$$

The power required is given by

$$P(t) = u(t)\dot{x}(t) = (ma - cA)at + ca^2t^3/2 \tag{2.2}$$

Note that the power required at time $t = t^*$ does not depend on the spring coefficient C

$$P^* = P(t^*) = ma\sqrt{2Aa}$$

Nevertheless, the time history $P(t)$ depends on the spring and will also affect the work.

Graphs of the function P/P^* are shown in Fig. 5 for two cases: when there is no spring (the dashed line) and when it is strong enough (the solid line) for a whole cycle of the process. Obviously, in the second and third quarters it yields $\dot{x} < 0$, which means deceleration.

The work in the first quarter of the first cycle, according to (1.5), is

$$W(t) = (ma - cA)at^2/2 + ca^2t^4/8 \tag{2.3}$$

Without the spring, $c = 0$, the work is always positive and results by the end of the first quarter of the first cycle $t = t^*$ in $W = maA$. During this time interval the power (2.2) is always positive, also and the element of work in the first interval defined by (2.8) is

$$\Delta W_1(t) = ma^2t^2/2, \quad 0 \leqslant t \leqslant t^*$$

With a weak spring $c \leqslant ma/A$ the first interval remains the same and the work reduces to

$$W(t^*) = maA - cA^2/2 \tag{2.4}$$

In particular, for $c = ma/A$ the work is reduced by 50%. The element of work $\Delta W_1(t)$ for that first interval is identical with (2.3).

For a strong spring the first cycle shows a deceleration and an acceleration phase as shown in Fig. 6. The time t_1 when the power vanishes follows from (2.2) as

$$t_1 = \sqrt{2(cA - ma)/(ca)} \tag{2.5}$$

The elements of work of the first interval is obtained from (2.3) where

$$\Delta W_1(t_1) = -(ma - cA)^2/(2c)$$

The element of work for the second interval follows from definition (1.7) and is

$$\Delta W_2(t) = W(t) - \Delta W_1(t)$$

where

$$\Delta W_2(t_2) = maA - cA^2/2 + (ma - cA)^2/(2c)$$

Both elements of work are shown in Fig. 6 using definition (1.8).

For the fully active actuator, the total work, according to (1.9), amounts to

$$\frac{W(t_2)}{maA} = -1 + \tfrac{1}{2}\gamma + \frac{1}{\gamma}, \quad \gamma = \frac{cA}{ma}, \quad t_2 = t^* \tag{2.6}$$

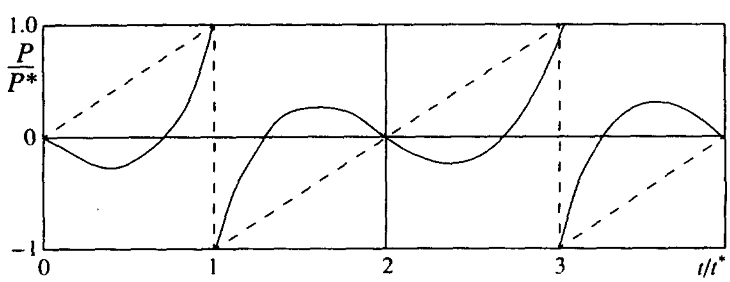

Fig. 5

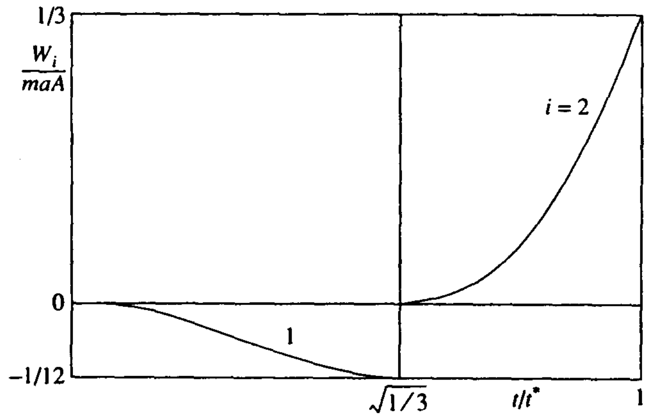

Fig. 6

The optimal spring coefficient is obtained by differentiating (2.6). We obtain $\gamma = \sqrt{2}$, resulting in an additional energy saving of 9%.

In a final step the maximum required power in the first interval has to be checked. As was pointed out, the power supply affects the investment costs. The time for the maximum power t_M follows from (2.2) by differentiation and amounts to

$$t_M = t_1 / \sqrt{3}$$

which is close to the middle of the first interval, see (2.5). Thus, the maximum power in the optimal case $\gamma = \sqrt{2}$ is

$$P(t_M) = -2^{3/4}\left(\frac{\sqrt{2}-1}{3}\right)^{3/2} ma\sqrt{2aA} = -0.086ma\sqrt{2Aa}$$

which is considerably less than the power (2.3).

In the second quarter of the cycle $t^* \leqslant t \leqslant 2t^*$ the following relations are obtained due to the bang-bang control

$$x(t) = \sqrt{2Aa}(t-t^*) - a(t-t^*)^2/2$$
$$\dot{x}(t) = \sqrt{2Aa} - a(t-t^*), \quad \ddot{x}(t) = -a$$
$$u(t) = -ma + c\sqrt{2Aa}(t-t^*) - ca(t-t^*)^2/2$$

Due to the symmetry of the motion in the four quarter cycles, for the energy and power evaluation no new results are obtained. Therefore, the corresponding equations will not be presented.

3. NON-LINEAR SPRING STORAGE

Compared to the harmonic motion presented in Section 1, the energy savings by a linear spring are less for an arbitrary prescribed motion. However, the energy storage capacity of systems with an arbitrary prescribed motion may be increased with a non-linear characteristic. For this purpose once again the oscillator (1.1) with prescribed motion (2.1) is considered.

For the design of a non-linear characteristic $n = n(x)$ the first quarter of a cycle is treated. The actuator force is obtained from inverse dynamics as

$$u(t) = ma + cn(x)$$

where c is a constant. Some typical characteristics are linear ($n(x) = x$), cubic progressive ($n(x) = \delta x + \varepsilon x^3$) and cubic regressive springs ($n(x) = \alpha x - \beta x^3$). To specify the shape coefficients α, β, δ and ε additional assumption for $n(x)$ are made.

For a regressive cubic spring it is assumed that

$$n(A) = \alpha A - \beta A^3 = A, \quad n'(A) = \alpha - 3\beta A^2 = 0$$

with $\alpha = 3/2$ and $\beta = 1/2 A^2$. For a progressive cubic spring

$$n(A) = \delta A + \varepsilon A^3 = A, \qquad n'(0) = \delta = 0$$

with $\delta = 0$ and $\varepsilon = 1/A^2$.

It then follows for $c = ma/A$ for the actuator force in the case of cubic progressive and cubic regressive springs respectively

$$u_1(t) = ma\left(\frac{3}{2}\frac{a}{A}t^2 - \frac{3}{4}\frac{a^2}{A^2}t^4 + \frac{1}{8}\frac{a^3}{A^3}t^6\right) \tag{3.1}$$

$$u_2(t) = ma\left(\frac{3}{8}\frac{a^2}{A^2}t^4 - \frac{1}{16}\frac{a^3}{A^3}t^6\right)$$

The functions (3.1) are represented in Fig. 7 (dot-and-dash and dashed line respectively), where solid line corresponds to linear characteristic. Obviously, the actuator force with the degressive characteristic is the smallest one what it also true for the power since the velocity does not change its sign in the first quarter of the cycle.

With the above assumptions, the power in the first quarter of the cycle follows by definition from (2.1) and (3.1) as

$$P(t) = ma\left(\frac{3}{8}\frac{a^3}{A^2}t^5 - \frac{1}{16}\frac{a^4}{A^3}t^7\right) \tag{3.2}$$

Integrating (3.2) in the first interval, which is identical with the first quarter of the cycle, we get

$$W(t) = ma\left(\frac{3}{48}\frac{a^3}{A^2}t^6 - \frac{1}{144}\frac{a^4}{A^3}t^8\right) \tag{3.3}$$

At the end point $t_1 = t^*$ formula (3.3) becomes

$$W(t_1) = \frac{7}{18} maA \tag{3.4}$$

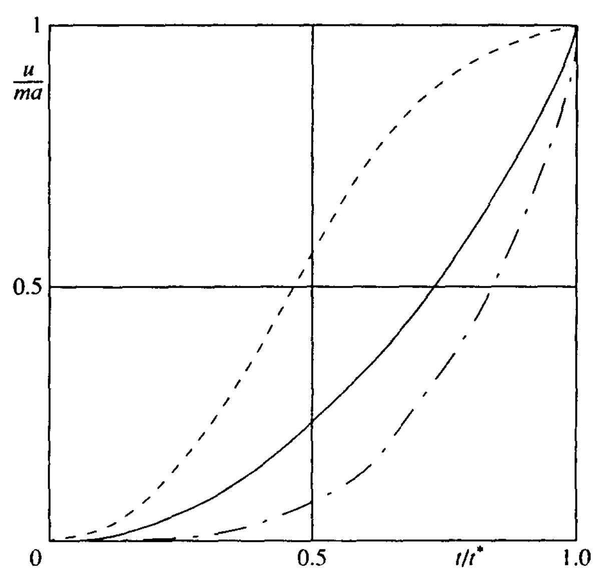

Fig. 7

Compared with (2.5) there is an additional 11% energy saving due to the non-linear spring, which is very remarkable.

This study was performed for the special case of the upper limit of the linear weak spring $c = ma/A$ to show the fundamental phenomena of non-linear energy storage. This approach may also be combined with a strong linear spring and additional energy savings can be expected. However, a detailed analysis is beyond the scope of this paper.

4. CONCLUDING REMARKS

Rheonomic constraints, or prescribed motions for inverse dynamic control, respectively, result in a high power and energy demand. Elastic springs offer energy savings which may easily reach 50% or more. The design parameters for the storage elements are the stiffness of linear springs and the characteristics of non-linear springs, respectively, which are recommended for all non-harmonic motions. The power aspects of inverse dynamics control systems or rheonomic constraints has been developed for actively controlled oscillators with one degree of freedom. This principle can be extended to multibody systems with an arbitrary number of bodies. Then, once again the concept of the element of work is available, based on the power requirement of each actuator in the system.

REFERENCES

1. FU, K. S., GONZALEZ, R. C. and LEE, C. S. G., *Robotics: Control, Sensing, Vision, and Intelligence*. McGraw-Hill, New York, 1987.
2. FALB, P.L. and WOLOVICH, W. A., Decoupling in the design and synthesis of multivariable control systems. *IEEE Trans. on Automat. Control*, 1967, **12**, 6, 651–655.
3. ASADA, H. and SLOTINE, J.-J.E., *Robot Analysis and Control*. Wiley, New York, 1986.
4. CRAIG, J. J., *Adaptive Control of Mechanical Manipulators*. Addison-Wesley, Reading, MA, 1988.
5. CRAIG, J. J., *Introduction to Robotics: Mechanics and Control*. Addison-Wesley, Reading, MA, 1985.
6. KOIVO, A. J., *Fundamentals for Control of Robotic Manipulator*. Wiley, New York, 1989.
7. SPONG, M. W. and VIDYASAGAR, M., *Robot Dynamics and Control*. Wiley, New York, 1989.
8. WALDRON, K. J., Some thoughts on the design of power systems for legged vehicles. *Adv. Multibody Systems and Mechatronics*. Gerhard-Mercator-Universitaet Duisburg, Duisburg, 1999.
9. RUMYANTSEV, V. V., The stability of the motion of nonholonomic systems. *Prikl. Mat. Mekh.*, 1967, **31**, 2, 260–271.
10. RUMYANTSEV, V. V., On the motion of controllable mechanical systems. *Prikl. Mat. Mekh.*, 1976, **40**, 5, 771–781.
11. RUMYANTSEV, V. V., The dynamics of Lagrangian rheonomic systems with constraints. *Prikl. Mat. Mekh.*, 1984, **48**, 4, 540–550.
12. RUMYANTSEV, V. V., General equations of classical dynamics. *Prikl. Mat. Mekh.*, 1996, **60**, 6, 917–928.
13. SCHIEHLEN, W., Multibody system dynamics: roots and perspectives. *Multibody Systems Dynamics*. 1997, **1**, 2, 149–188.
14. SCHIEHLEN, W. O., Power aspects of inverse dynamics control systems. Institut B fur Mechanik, Institutsbericht IB-35, Stuttgart, 2000.

Translated by the author